be no completely redundant controls, the algorithm directly identifies each of the $m(m-1)$ facets of the attainable moment subset without searching or iteration. Since there are $2^{(m-2)}m!/[(2!(m-2)!]$ faces in the subset that are potentially on the boundary, the computational savings of this algorithm over a brute-force search can be impressive. Further computational savings can be achieved by noting that the vertices of a particular facet are always shared by other facets, and their coordinates need not be recalculated for each new facet.

For example, with 20 controls, there are almost 50 million facets in the subset of constrained controls, and the image of each of these is potentially a facet in the subset of attainable moments. Since facets are generated in pairs, the algorithm will identify the 380 facets out of the 50 million candidates in 190 passes. Whereas each facet has four vertices, there are not $4 \times 380 = 1520$ separate vertices to be calculated, but only $m(m-1)+2 = 382$ vertices.

The ability to calculate attainable moment subsets rapidly and reliably has two important benefits. First, one need not precalculate the subsets for use in allocating the controls in flight. Instead, aerodynamic tables of control effectiveness may be interpolated on-the-fly and new moment subsets generated for any flight condition encountered. Second, the attainable moment subset may be recalculated immediately following the identification of a control effector failure. The failed control effector is simply removed from the problem, both in the B matrix and in the control vector. If the control law used in the flight control system deals with generalized controls (three orthogonal moment generators) and leaves the actual allocation of the surfaces to the allocator, then the control law proper will be oblivious to the failure of a control effector, so long as the remaining effectors are capable of generating the moments implied by the generalized controls.

The determination of the attainable moment subset only solves half of the control allocation problem. The other half, the actual allocation of controls based on the moment subset description, is not a difficult task. Following this discussion, we conclude that real-time implementation of the algorithm presented in this Note, as part of a multicontrol effector control system, is both practical and desirable.

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Analysis of Nonlinear Equations by Robust Stability Theory

Howard Musoff*
Charles Stark Draper Laboratory, Inc., Cambridge, Massachusetts 02139

Introduction

A PROPOSITION and its proof relating the determination of asymptotic stability of nonlinear differential equations to robust stability theory for structured uncertainties are introduced. An example illustrating the resulting technique is presented.

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*Principal Member, Technical Staff. Member AIAA.

Proposition

The nonlinear state equation is written

$$\dot{x} = A(x)x \tag{1}$$

where the elements of $A(x)$ range according to all possible values of the components of x.

If required, the elements of $A(x)$ can always be bounded with finite values by a suitable change in the time scale given by

$$\frac{d\tau}{dt} = g(x) \tag{2}$$

where t is the original time scale, τ is the transformed scale, and

$$g(x) \geq 1 \tag{3}$$

The resulting system of equations

$$\frac{dx}{d\tau} = \frac{A(x)}{g(x)}x \tag{4}$$

has bounded coefficients of x and is asymptotically stable at the equilibrium point of $\mathbf{0}$ if Eq. (1) is and vice versa.

The elements of the bounded matrix can be considered to belong to interval sets. Thus for a bounded $B(x)$

$$B_{ij} \in [\min B_{ij}, \max B_{ij}] \tag{5}$$

where the brackets denote an interval set.

The proposition for determining asymptotic stability of the original nonlinear equations is that robust stability theory (i.e., using the classical Routh-Hurwitz array with interval arithmetic[1]) for structured uncertainties can also be applied to those equations (derived from the nonlinear equations) that contain interval sets of the type (5).

Proof of Proposition

If the original differential equation (1) is asymptotically stable and has a suitable autonomous Lyapunov positive-definite function $v(x)$ in the domain $0 < x < x_m$, then

$$\frac{\partial v}{\partial x}\frac{dx}{dt} = \frac{\partial v}{\partial x}A(x)x \tag{6}$$

$$\frac{\partial v}{\partial x}A(x)x = w(x) \tag{7}$$

$$w(x) < 0 \tag{8}$$

and $w(x)$ is negative definite according to Lyapunov's direct method.[2]

After the suitable time scale change given by Eq. (4) we have

$$\frac{\partial v}{\partial x}\frac{dx}{d\tau} = \frac{\partial v}{\partial x}\frac{A(x)}{g(x)}x \tag{9}$$

$$\frac{\partial v}{\partial x}\frac{A(x)}{g(x)}x = \frac{w(x)}{g(x)} < 0 \tag{10}$$

because of Eq. (3).

Thus the equations with the new time scale τ are asymptotically stable if the original equations are and vice versa.

Therefore, we assume in the domain

$$0 < x < x_m \tag{11}$$

that suitable time scale changes have been made to yield

$$\frac{dx}{\partial\tau} = B(x)x \tag{12}$$

where the elements of $B(x)$ are bounded by finite values. That is,

$$B(x) \in [B_l, B_u] \tag{13}$$

The interval set denoted by the brackets is an interval set of matrices where B_l is a matrix of lower bounds on the elements of $B(x)$ and B_u is a matrix of upper bounds on the elements of $B(x)$.

Using the basic properties of interval sets[3] we see that

$$\frac{\partial v}{\partial x} B(x)x \in \left[\frac{\partial v}{\partial x} B_l x, \frac{\partial v}{\partial x} B_u x \right] \tag{14}$$

$$\left[\frac{\partial v}{\partial x} B_l x, \frac{\partial v}{\partial x} B_u x \right] \triangleq [w_l(x), w_u(x)] \tag{15}$$

with

$$w_l(x) < 0 \tag{16}$$

$$w_u(x) < 0 \tag{17}$$

so that

$$[w_l(x), w_u(x)] < 0 \tag{18}$$

and the equivalent linear system with the structured uncertainties given by the interval matrices $[B_l, B_u]$ will always be stable.

At this point we have proven that if the original equations are asymptotically stable, then the resulting structured uncertainty equations will be asymptotically stable. However, can the structured uncertainty linear equations indicate stability if the original equations are not stable? We can use the Lyapunov parallel theorems[2] to show how the structured uncertainty linear equations will not be stable if the original nonlinear equations are not stable and thereby answer the preceding question in the negative. This is done by repeating the above proof with again a positive-definite Lyapunov function $v(x)$ but with $w(x)$ being positive definite,

$$w(x) > 0 \tag{19}$$

for the original unstable equations instead of negative definite.

Limitation of Proof

The proof made use of the assumption of existence of a positive-definite autonomous function $v(x)$. It is by no means guaranteed that such an autonomous function exists, although existence of the necessary positive-definite but nonautonomous function can be shown to exist [i.e., existence of the required $v(x, t)$ with v dependent on time as well as on x can be proven[2]].

Example

An example illustrating the usefulness of the technique is as follows.

Gibson[4] presents the example of a set of nonlinear equations

$$\dot{x}_1 = x_2 \tag{20}$$

$$\dot{x}_2 = -x_1^3 - x_2 \tag{21}$$

whose asymptotic stability can be proven with the aid of a fairly complicated variable gradient method for generating Lyapunov functions. The nonobvious Lyapunov function arrived at is

$$v = \tfrac{1}{2}x_1^4 + \tfrac{1}{2}x_1^2 + x_1x_2 + x_2^2 \tag{22}$$

with the resulting negative-definite function $w(x)$ being

$$w = -x_1^4 - x_2^2 < 0, \qquad x_1, x_2 \neq 0 \tag{23}$$

Using our technique, we make the time scale change

$$\frac{d\tau}{dt} = 1 + x_1^2 \tag{24}$$

so that

$$\frac{dx_1}{d\tau} = \alpha x_2 \tag{25}$$

$$\frac{dx_2}{d\tau} = -(1-\alpha)x_1 - \alpha x_2 \tag{26}$$

where

$$\alpha = 1/\left(1 + x_1^2\right) \tag{27}$$

$$0 < \alpha < 1 \tag{28}$$

The characteristic equation for (25) and (26) is

$$s^2 + \alpha s + (1-\alpha)s = 0 \tag{29}$$

which shows asymptotic stability[1] for the conditions of Eq. (28).

Conclusions

A proposition relating nonlinear stability analysis to stability theory for linear differential equations having structured uncertainties was proved and illustrated by an example. The stability analysis of nonlinear differential equations is greatly simplified by applying the proposition because the necessity of guessing a Lyapunov function that works for each particular case can be eliminated, as illustrated by the example.

The proof that was presented strengthens confidence in the use of the resulting technique for solving applicable problems, albeit a limitation of the proof is the assumption of existence of an autonomous Lyapunov function for the nonlinear differential equations.

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Multiple Optimal Solutions for Structural Control Using Genetic Algorithms with Niching

K. KrishnaKumar,* R. Swaminathan,†
and L. Montgomery†
University of Alabama, Tuscaloosa, Alabama 35487

Introduction

THE interaction between the placement of actuators and sensors and the optimal control system synthesis poses a unique challenge to the problem of optimal structural control. Studies conducted in the past have generally examined the structural control problem to consist of two separate problems: 1) placement of actuators/sensors to minimize some criteria and 2) syntheses of feedback control systems to suppress structural vibration, station keeping, attitude control, etc. For example, Refs. 1–3 use a linear quadratic regulator (LQR) technique to synthesize a feedback controller and then use the resulting control system for the computation of a performance index (PI for actuator/sensor placement) based on control effort. The resulting strategy, to a large extent, depends on the characteristics of the LQR. In Ref. 4, pole-positioning techniques are used with an optimal energy formulation. Here also, first the closed-loop poles are chosen and then the actuators are placed to minimize the

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*Assistant Professor of Aerospace Engineering, Department of Aerospace Engineering. Senior Member AIAA.

†Graduate Student of Aerospace Engineering, Department of Aerospace Engineering.